The influence of helicity on scaling regimes in the extended Kraichnan model

2006 J. Phys. A: Math. Gen. 39 7913

(http://iopscience.iop.org/0305-4470/39/25/S08)

Institute of Physics Publishing | Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **39** (2006) 7913–7926 doi:10.1088/0305-4470/39/25/S08

The influence of helicity on scaling regimes in the extended Kraichnan model

O G Chkhetiani[1], M Hnatich[2,3], E Jurčišinová[2,4], M Jurčišin[2,5], A Mazzino[6] and M Repašan[2]

[1] Space Research Institute, Profsoyuznaya 84/32, 117997 Moscow, Russian Federation
[2] Institute of Experimental Physics, Slovak Academy of Sciences, Watsonova 47, 04001, Košice, Slovakia
[3] Department of Mathematics, Faculty of Civil Engineering, Technical University Vysokoškolská 4, 040 01 Košice, Slovakia
[4] Laboratory of Information Technologies, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russian Federation
[5] N N Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russian Federation
[6] Department of Physics, University of Genova, National Institute of Nuclear Physics, Genova Section, via Dodecaneso 33, I-16146 Genova, Italy

E-mail: hnatic@saske.sk

Received 1 December 2005, in final form 10 February 2006
Published 7 June 2006
Online at stacks.iop.org/JPhysA/39/7913

Abstract
We have investigated the advection of a passive scalar quantity by an incompressible helical turbulent flow in the framework of an extended Kraichnan model. Turbulent fluctuations of velocity field are assumed to have the Gaussian statistics with zero mean and defined noise with finite time correlation. Actual calculations have been done up to two-loop approximation in the frame of field-theoretic renormalization group approach. It turned out that space parity violation (helicity) of a turbulent environment does not affect anomalous scaling which is a peculiar attribute of the corresponding model without helicity. However, stability of asymptotic regimes, where anomalous scaling takes place, strongly depends on the amount of helicity. Moreover, helicity gives rise to the turbulent diffusivity which has been calculated in the one-loop approximation.

PACS numbers: 47.27.−i, 47.10.+g, 05.10.Cc

1. Introduction

During the last decade much attention has been paid to the inertial range of fully developed turbulence which contains wave numbers larger than those that pump the energy into the system and smaller enough than those that are related to the dissipation processes [1]. Grounding of the inertial range turbulence has been created in the well-known Kolmogorov–Obukhov (KO) phenomenological theory (see, e.g., [1–3]). One of the main problems in the modern theory of fully developed turbulence is to verify the validity of the basic principles of the KO theory and their consequences within the framework of a microscopic model. Recent experimental and theoretical studies indicate possible deviations from the celebrated Kolmogorov scaling exponents. The scaling behaviour of the velocity fluctuations with exponents whose values are different from the Kolmogorov ones, is called anomalous and is usually associated with the intermittency phenomenon. The deviations, referred to as 'anomalous' or 'non-dimensional' scaling, manifest themselves in singular (arguably power-like) dependence of correlation or structure functions on the distances and the integral (external) turbulence scale L. The corresponding exponents are certain nontrivial and nonlinear functions of the order of the correlation function, the phenomenon referred to as 'multiscaling'.

Although the theoretical description of the fluid turbulence on the basis of the 'first principles', i.e., on the stochastic Navier–Stokes (NS) equation [1] remains essentially an open problem, considerable progress has been achieved in understanding simplified model systems that share some important properties with the real problem. The crucial role in these studies is played by the models of advected passive scalar field [5]. A simple model of a passive scalar quantity advected by a random Gaussian velocity field, white in time and self-similar in space, the so-called Kraichnan's rapid-change model [6], is an example. There, for the first time, the anomalous scaling was established on the basis of a microscopic model [8], and the corresponding anomalous exponents were calculated within controlled approximations [9, 10] (see also review [4] and references therein).

The greatest stimulation to study the simple models of passive advection not only of scalar fields but also of vector fields (e.g., weak magnetic field) is related to the fact that even simplified models with given Gaussian statistics of the so-called 'synthetic' velocity field describe a lot of features of anomalous behaviour of genuine turbulent transport of some quantities (as heat or mass) observed in experiments (see, e.g. [7–11] and references cited therein).

The term 'anomalous scaling' reminds of the critical scaling in models of equilibrium phase transitions. In those, the field theoretic methods were successfully employed to establish the existence of self-similar (scaling) regimes and to construct regular perturbative calculational schemes (the famous ϵ expansion and its relatives) for the corresponding exponents, scaling functions, ratios of amplitudes etc (see, e.g. [12, 13]). Here and below, by 'field theoretic methods' we mean diagrammatic and functional techniques, renormalization theory and renormalization group, composite operators, operator-product expansion and so on [13].

The feature specific to the theory of turbulence and simplified models associated with it is the existence in the corresponding field theoretical models of the composite operators with *negative* scaling (critical) dimensions. Such operators, termed 'dangerous' in [14–18], give rise to the anomalous scaling, i.e., the singular dependence on the infrared (IR) scale L with certain nonlinear anomalous exponents.

Important advantages of the RG approach are its universality and calculational efficiency: regular systematic perturbation expansion for the anomalous exponents was constructed which is similar to the well-known ϵ-expansion in the theory of phase transitions, and the exponents were calculated in the first order of expansion for passively advected vector fields [19, 20] and in the second [14] and third [16] orders of that expansion for scalar fields. Furthermore, the RG approach is not related only to the rapid-change model and can also be applied to the case with finite correlation time, anisotropy, the space parity violation and, moreover, non-Gaussian advecting field [18].

The solution proceeds in two main stages. In the first stage, the multiplicative renormalizability of the corresponding field theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behaviour of the latter on their UV argument (r/l) (l is the internal length) for $r \gg \ell$ and any fixed (r/L) (L is an outer length) is given by IR stable fixed points of those equations. It involves some 'scaling functions' of the IR argument (r/L) whose form is not determined by the RG equations. In the second stage, their behaviour at $r \ll L$ is found from the operator product expansion within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite operators, which give rise to an infinite family of independent scaling exponents (and hence to multiscaling). Of course, both these stages (and thus the phenomenon of multiscaling) have long been known in the RG theory of critical behaviour. The distinguishing feature, specific to the models of turbulence is the existence of composite operators with the afore-mentioned *negative* critical dimensions. Their contributions to the operator product expansion diverge as $(r/L) \to 0$. In the models of critical phenomena, nontrivial composite operators have always positive dimensions and determine only corrections (vanishing as $(r/L) \to 0$) to the leading terms (finite for $(r/L) \to 0$) in the scaling functions.

The existence of regular perturbation schemes and accurate numerical simulations allows one to discuss, for the example of the rapid-change model and its descendants, the issues that are interesting within the general context of fully developed turbulence: universality and saturation of anomalous exponents, effects of compressibility, anisotropy and pressure, persistence of the large-scale anisotropy and hierarchy of anisotropic contributions and so on. Moreover, it is interesting and important to study the helicity (violation of space parity) effects because many turbulence phenomena are directly influenced by them.

In [18] the problem of a passive scalar advected by the Gaussian self-similar velocity field with finite correlation time [21] was studied by the field theoretic RG method. There, the systematic study of the possible scaling regimes and anomalous behaviour was present at the one-loop level. The two-loop corrections to the anomalous exponents were obtained in [22]. It was shown that the anomalous exponents are nonuniversal as a result of their dependence on a dimensionless parameter, the ratio of the velocity correlation time and turnover time of a scalar field.

In what follows, we shall continue with the investigation of this model from the point of view of the influence of helicity on the scaling regimes and the anomalous exponents within the two-loop approximation.

Helicity is defined as the scalar product of velocity and vorticity and its nonzero value expresses mirror symmetry breaking of a turbulent flow. It plays a significant role in the processes of magnetic field generation in electrically conductive fluid [23, 24] and represents one of the most important characteristics of large-scale motions as well [25, 26]. Despite this fact the role of the helicity in hydrodynamical turbulence is not completely clarified up to now.

The Navier–Stokes equations conserve kinetic energy and helicity in inviscid limit. The presence of two quadratic invariants leads to the possibility of appearance of a double cascade. It means that cascades of energy and helicity take place in different ranges of wave numbers analogously to the two-dimensional turbulence and/or the helicity cascade appears concurrently to the energy one in the direction of small scales [27, 28]. Particularly, a helicity cascade is closely connected with the existence of exact relation between triple and double correlations of velocity known as the '2/15' law analogously to the '4/5' Kolmogorov law [29]. The afore-mentioned scenarios of turbulent cascades corresponding to [27] differ from each other by spectral scaling. Theoretical arguments given by Kraichnan [30] and results of numerical calculations of Navier–Stokes equations [31] support the scenario of concurrent cascades. The appearance of helicity in a turbulent system leads to constraint of nonlinear cascade to the small scales. This phenomenon was firstly demonstrated by Kraichnan [30] within the modelling problem of statistically equilibrium spectra and later in numerical experiments.

Turbulent viscosity and diffusivity, which characterize influence of small-scale motions on heat and momentum transport, are basic quantities investigated in the theoretic and applied models. The constraint of direct energy cascade in helical turbulence has to be accompanied by a decrease of turbulent viscosity. However, no influence of helicity on turbulent viscosity was found in some works [32, 33], but, we want to stress that calculations were made only in the lowest (one-loop) nontrivial approximation. A similar situation is observed for turbulent diffusivity in helical turbulence. Although the modelling calculations demonstrate intensification of turbulent transfer in the presence of helicity [34], direct calculation of diffusivity does not confirm this effect [35, 36]. Helicity is the pseudoscalar quantity; hence, it can be easily understood that its influence appears only in quadratic and higher terms of perturbation theory or in the combination with other pseudoscalar quantities (e.g., large-scale helicity). Really, simultaneous consideration of memory effects and second order approximation indicate an effective influence of helicity on turbulent viscosity [37] and turbulent diffusivity [34, 38, 39] already in the limit of small or infinite correlation time.

Helicity, as we shall see below, does not affect known results in the one-loop approximation and, therefore, it is necessary to turn to the second order (two-loop) approximation to be able to analyse possible consequences. It is also important to say that in the framework of the classical Kraichnan model, i.e., the model of passive advection by the Gaussian velocity field with δ-like correlations in time, it is not possible to study the influence of the helicity because all potentially 'helical' diagrams are identically equal to zero at all orders in the perturbation theory. In this sense, the investigation of the helicity in the present model can be considered as the first step to analyse the helicity in genuine turbulence.

2. Field theoretic description of the model

The advection of a passive scalar field $\theta(x) \equiv \theta(t, \mathbf{x})$ in helical turbulent environment is described by the stochastic equation

$$\partial_t \theta + v_i \partial_i \theta = \nu_0 \Delta \theta + f, \tag{1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial \mathrm{x}_i$, ν_0 is the molecular diffusivity coefficient (hereafter all parameters with a subscript 0 denote bare parameters of unrenormalized theory; see below), $\triangle \equiv \partial^2$ is the Laplace operator, $v_i \equiv v_i(x)$ is the ith component of the divergence-free (owing to the incompressibility) velocity field $\mathbf{v}(x)$, and $f \equiv f(x)$ is an artificial the Gaussian random noise with zero mean and correlation function

$$\langle f(x) f(x') \rangle = \delta(t - t') C(\mathbf{r}/L), \qquad \mathbf{r} = \mathbf{x} - \mathbf{x}', \tag{2}$$

where L denotes an integral (outer) scale. It maintains the steady-state of the system but the detailed form of the function $C(\mathbf{r}/L)$ is unessential in our consideration. In spite of the fact that in real problems the velocity field $\mathbf{v}(x)$ satisfies the Navier–Stokes equation, in what follows, we suppose that the statistics of velocity field is given in the form of Gaussian distribution with zero mean and correlator

$$\langle v_i(x) v_j(x') \rangle = \int \frac{\mathrm{d}\omega\, \mathrm{d}^d k}{(2\pi)^{d+1}} P^{\rho}_{ij}(\mathbf{k}) D_v(\omega, k) \exp[-\mathrm{i}\omega(t - t') + \mathrm{i}\mathbf{k}(\mathbf{x} - \mathbf{x}')], \tag{3}$$

with

$$D_v(\omega, k) = \frac{D_0 k^{4-d-2\varepsilon-\eta}}{(\mathrm{i}\omega + u_0 \nu_0 k^{2-\eta})(-\mathrm{i}\omega + u_0 \nu_0 k^{2-\eta})}, \tag{4}$$

where $k = |\mathbf{k}|$, $D_0 = g_0 \nu_0^3$ is a positive amplitude factor, g_0 plays the role of the coupling constant of the model, an analogue of the coupling constant λ_0 in the $\lambda_0\varphi^4$ model of critical behaviour [12, 13]. In addition, g_0 is a formal small parameter of the ordinary perturbation theory. The positive exponents ε and η ($\varepsilon = O(\eta)$) are small RG expansion parameters, the analogues of the parameter $\varepsilon = 4 - d$ in the $\lambda_0\varphi^4$ theory. Thus we have a kind of double expansion model in the $\varepsilon - \eta$ plane around the origin $\varepsilon = \eta = 0$. The correlator (4) is directly related to the energy spectrum via the frequency integral [18]

$$E(k) \simeq k^{d-1} \int \mathrm{d}\omega\, D^v(\omega, k) \simeq \frac{g_0 \nu_0^2}{u_0} k^{1-2\varepsilon}. \tag{5}$$

Therefore, the coupling constant g_0 and the exponent ε describe the equal-time velocity correlator or, equivalently, energy spectrum. On the other hand, the constant u_0 and the second exponent η are related to the frequency $\omega \simeq u_0 \nu_0 k^{2-\eta}$ which characterizes the mode $\mathbf{k}$ [40]. Thus, in our notation, the value $\varepsilon = 4/3$ corresponds to the well-known Kolmogorov 'five-thirds law' for the spatial statistics of velocity field, and $\eta = 4/3$ corresponds to the Kolmogorov frequency. For completeness, we remain d-dependence in expressions (3) and (4) (d is the dimensionality of the $\mathbf{x}$ space), although, of course, when one investigates system with helicity the dimension of the $\mathbf{x}$ space must be strictly equal to three. To include helicity the transverse projector $P_{ij}^{\rho}(\mathbf{k})$ is taken in the form

$$P_{ij}^{\rho}(\mathbf{k}) = P_{ij}(\mathbf{k}) + H_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2 + \mathrm{i}\rho \epsilon_{ijl} \frac{k_l}{k}. \tag{6}$$

Here $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ represents the non-helical part of the total transverse projector $P_{ij}^{\rho}(\mathbf{k})$. On the other hand, $H_{ij}(\mathbf{k}) = i\rho\epsilon_{ijl} k_l / k$ mimics the presence of helicity in the flow. Thus, formally, the transition to the helical fluid corresponds to the breaking of spatial parity, and, technically, this is expressed by the fact that the correlation function is specified in the form of mixture of a true tensor and a pseudotensor. The tensor ε_{ijl} is completely antisymmetric tensor of rank 3 and the real parameter ρ characterizes the amount of helicity. Due to the requirement of positive definiteness of the correlation function the absolute value of ρ must be in the interval $|\rho| \in \langle 0, 1\rangle$. The nonzero helical part proportional to ρ physically expresses the existence of nonzero correlations $\langle \mathbf{v} \cdot \mathrm{rot}\, \mathbf{v} \rangle$.

The general model (3), (4) contains two important special cases: a rapid-change model limit when $u_0 \to \infty$ and $g_0' \equiv g_0 / u_0^2 = \mathrm{const}$, $D_v(\omega, k) \to g_0' \nu_0 k^{-d-2\varepsilon+\eta}$, and a quenched (time-independent or frozen) velocity field limit which is defined by $u_0 \to 0$ and $g_0'' \equiv g_0/u_0 = \mathrm{const}$, $D_v(\omega, k) \to g_0'' \nu_0^2 \pi \delta(\omega) k^{-d+2-2\varepsilon}$, which is similar to the well-known models of the random walks in random environment with long range correlations (see, e.g. [41, 42]).

Using Martin–Siggia–Rose mechanism [43–46] the stochastic problem (1)–(4) can be treated as a field theory with action functional

$$S(\theta, \theta', \mathbf{v}) = \theta' D_\theta \theta'/2 + \theta'[-\partial_t + \nu_0 \triangle - (v_i \partial_i)]\theta - \mathbf{v} D_v^{-1} \mathbf{v}/2, \tag{7}$$

where θ' is an auxiliary scalar field, and D_θ and D_v are correlators (2) and (3), respectively. In the action (7) all the required integrations over $x = (t, \mathbf{x})$ and summations over the vector indices are understood. The first four terms in equation (7) represent the Dominicis–Jansen-type action for the stochastic problem (1), (2) at fixed $\mathbf{v}$, and the last term represents the Gaussian averaging over $\mathbf{v}$.

The model (7) corresponds to a standard Feynman diagrammatic technique with the bare propagators $\langle \theta\theta' \rangle_0$ and $\langle v_i v_j \rangle_0$ (in the momentum–frequency representation)

$$\langle \theta(\omega, \mathbf{k}) \theta'(\omega, \mathbf{k}) \rangle_0 = \frac{1}{-\mathrm{i}\omega + \nu_0 k^2}, \qquad \langle v_i(\omega, \mathbf{k}) v_j(\omega, \mathbf{k}) \rangle_0 = P_{ij}^{\rho}(\mathbf{k}) D_v(\omega, k), \tag{8}$$

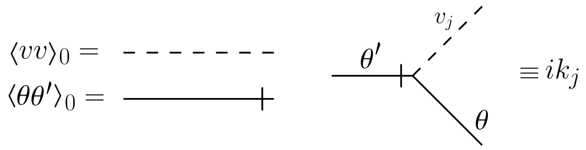

Figure 1. (Left) Graphical representations of the needed propagators of the model. (Right) The triple (interaction) vertex of the model. The momentum **k** is entering into the vertex via field θ'.

where $D_v(\omega, k)$ is given directly by (4). In the Feynman diagrams, these propagators are represented by the lines which are shown in figure 1 (the end with a slash in the propagator $\langle\theta\theta'\rangle_0$ corresponds to the field θ', and the end without a slash corresponds to the field θ). The triple vertex (or interaction vertex) $-\theta' v_j \partial_j \theta = \theta' v_j V_j \theta$, where $V_j = \mathrm{i}k_j$ (in the momentum–frequency representation), is presented in figure 1, where the momentum **k** is flowing into the vertex via the auxiliary field θ'.

3. Renormalization group analysis

The model (7) is logarithmic for $\epsilon = \eta = 0$ (the coupling constant g_0 is dimensionless) and, in this case, possible ultraviolet (UV) divergences have the form of poles in various linear combinations of ε and η in the correlation functions. The reader can find a detailed description of how to apply the procedure of elimination of UV divergencies and use the technique of renormalization group in the theory of developed turbulence in books [13, 15]. Using the standard analysis of quantum field theory one finds that all divergences can be removed by the only counterterm of the form $\theta'\triangle\theta$ [18]. Thus, the model is multiplicatively renormalizable, which is expressed explicitly in the multiplicative renormalization of the parameters g_0, u_0 and ν_0 in the form

$$\nu_0 = \nu Z_\nu, \qquad g_0 = g\mu^{2\varepsilon+\eta} Z_g, \qquad u_0 = u\mu^\eta Z_u. \tag{9}$$

Here the dimensionless parameters g, u and ν are the renormalized counterparts of the corresponding bare ones, μ is the renormalization mass (a scale setting parameter), an artefact of dimensional regularization. Newly introduced quantities $Z_i = Z_i(g, u; d, \rho; \varepsilon, \eta) = Z_i(g, u; d, \rho; \varepsilon)$ are renormalization constants (note that if ρ is nonzero then $d = 3$) and, in general, contain poles of linear combinations of ε and η. However, as detailed analysis shows, to obtain all important quantities as the γ-functions, β-functions, coordinates of fixed points, and the critical dimensions, the knowledge of the renormalization constants for the special choice $\eta = 0$ is sufficient up to two-loop approximation (see details in [18]).

The renormalized action functional has the following form:

$$S_R(\theta, \theta', \mathbf{v}) = \theta' D_\theta \theta'/2 + \theta'[-\partial_t + \nu Z_1 \triangle - (v\partial)]\theta - \mathbf{v} D_v^{-1} \mathbf{v}/2, \tag{10}$$

where the correlator D_v is written in renormalized parameters. By comparison of the renormalized action (10) with definitions of the renormalization constants Z_i, $i = g, u, \nu$ (9) one comes to the relations among them:

$$Z_\nu = Z_1, \qquad Z_g = Z_\nu^{-3}, \qquad Z_u = Z_\nu^{-1}. \tag{11}$$

The second and third relations are consequences of the absence of the renormalization of the term with D_v in renormalized action (10). The parameter ρ as the fields θ, θ', **v** are not renormalized; therefore $Z_\rho = Z_\theta = Z_{\theta'} = Z_\mathbf{v} = 1$.

The issue of interest is, in particular, the behaviour of the equal-time structure functions

$$S_n(r) \equiv \langle[\theta(t, \mathbf{x}) - \theta(t, \mathbf{x}')]^n\rangle, \qquad r \equiv |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'| \tag{12}$$

in the inertial range, specified by the inequalities $l \ll r \ll L$ (l is the internal length). Here parentheses $\gg$ mean the functional average over the fields $\theta, \theta', \mathbf{v}$ with the weight $\exp(S_R)$. In the isotropic case, the odd functions S_{2n+1} vanish, while for S_{2n} simple dimensionality considerations give

$$S_{2n}(r) = \nu_0^{-n} r^{2n} R_{2n}(r/l, r/L, g_0, u_0, \rho), \tag{13}$$

where R_{2n} are some functions of dimensionless variables. In principle, they can be calculated within the ordinary perturbation theory (i.e., as series in g_0), but this is not useful for studying inertial-range behaviour: the coefficients are singular in the limits $r/l \to \infty$ and/or $r/L \to 0$, which compensate the smallness of g_0, and in order to find correct infrared behaviour we have to sum the entire series. The desired summation can be accomplished using the field theoretic renormalization group (RG) and operator product expansion (OPE)(see [14, 16, 18] for details).

The RG analysis consists of two main stages. In the first stage, the multiplicative renormalizability of the model is demonstrated and the differential RG equations for its correlation (structure) functions are obtained. The asymptotic behaviour of the functions like (12) for $r/l \gg 1$ and any fixed r/L is given by IR stable fixed points g_*, u_* (see below) of the RG equations and has the form

$$S_{2n}(r) = \nu_0^{-n} r^{2n} (r/l)^{-\gamma_n} R_{2n}(r/L, \rho), \quad r/l \gg 1 \tag{14}$$

with certain, as yet unknown, scaling functions $R_{2n}(r/L, \rho) \equiv R_{2n}(1, r/L, g_*, u_*, \rho)$. In the theory of critical phenomena [12, 13] the quantity $\Delta[S_{2n}] \equiv -2n + \gamma_n$ is termed the critical dimension, and the exponent γ_n, the difference between the critical dimension $\Delta[S_{2n}]$ and the canonical dimension $-2n$, is called the anomalous dimension.

In the second stage, the small r/L behaviour of the functions $R_{2n}(r/L, \rho)$ is studied within the general representation (14) using the operator product expansion (OPE). It shows that, in the limit $r/L \to 0$, the functions $R_{2n}(r/L, \rho)$ have the asymptotic forms

$$R_{2n}(r/L) = \sum_F C_F(r/L)(r/L)^{\Delta_n}, \tag{15}$$

where C_F are coefficients regular in r/L. In general, the summation is implied over certain renormalized composite operators F with critical dimensions Δ_n. In case under consideration the leading operators F have the form $F_n = (\nabla_i\theta\nabla_i\theta)^n$.

We have performed the complete two-loop calculation of the critical dimensions of the composite operators F_n for arbitrary values of n, d, u and ρ and obtain them in the following form:

$$\Delta_n = \Delta_n^{(1)}\epsilon + \Delta_n^{(2)}\epsilon^2, \qquad \Delta_n^{(1)} = \frac{-n(n-2)(d-1)}{2(d-1)(d+2)}, \tag{16}$$

where $\Delta_n^{(1)}$ is critical dimension obtained in the one-loop approximation. Interesting technical details of these two-loop calculations will be present elsewhere.

Two-loop contribution $\Delta_n^{(2)}$ is rather cumbersome and can be found in [22]. The main and interesting result consists in the fact that although separated two-loop Feynman graphs of operators F_n strongly depend on the helicity parameter ρ, such dependence disappears in their sum, which gives rise to the critical dimensions Δ_n. We can conclude that in the two-loop approximation anomalous scaling with *negative* exponents (16) is not affected by the existence of nonzero helical correlations $\langle \mathbf{v} rot \mathbf{v}\rangle$ in turbulent incompressible flow. It turns out,

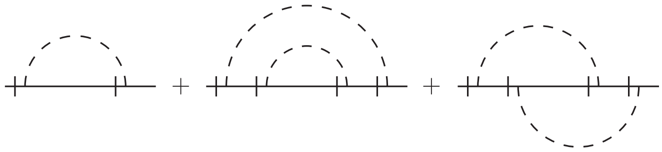

Figure 2. The one- and two-loops contributions to the self-energy operator Σ.

however, that region of stability of possible asymptotic regimes governed by fixed points of RG equations, where anomalous scaling takes place, and effective diffusivity strongly depends on ρ.

Let us analyse asymptotic regimes in detail. The structure functions and the other statistical averages of random fields θ, θ' satisfy linear differential RG equations with the linear differential operator $\mathcal{D}_{RG}$:

$$\mathcal{D}_{RG} = \mathcal{D}_{\mu} + \beta_{\mathrm{g}}(\mathrm{g}, u)\partial_{\mathrm{g}} + \beta_{u}(\mathrm{g}, u)\partial_{u} - \gamma_{\nu}(\mathrm{g}, u)\mathcal{D}_{\nu}. \tag{17}$$

Here $\mathcal{D}_x \equiv x\partial_x$ stands for any variable x and the RG functions (the β and γ functions) are given by well-known definitions and in our case, using the relations (11) for the renormalization constants, they acquire the following form:

$$\gamma_{\nu} \equiv \tilde{\mathcal{D}}_{\mu} \ln Z_{\nu}, \tag{18}$$

$$\beta_{\mathrm{g}} \equiv \tilde{\mathcal{D}}_{\mu}\mathrm{g} = \mathrm{g}(-2\varepsilon - \eta + 3\gamma_{\nu}), \tag{19}$$

$$\beta_{u} \equiv \tilde{\mathcal{D}}_{\mu}u = u(-\eta + \gamma_{\nu}). \tag{20}$$

The renormalization constant Z_{ν} is determined by the requirement that the response function $G \equiv \langle\theta\theta'\rangle$ must be UV finite when is written in renormalized variables. In our case it means that it has no singularities in the limit $\varepsilon, \eta \to 0$. The response function G is related to the self-energy operator Σ, which is expressed via Feynman graphs, by the Dyson equation. In frequency–momentum representation it has the following form:

$$G(\omega, \mathbf{p}) = \frac{1}{-\mathrm{i}\omega + \nu_0 p^2 - \Sigma(\omega, \mathrm{p})}. \tag{21}$$

Thus, Z_{ν} is found from the requirement that the UV divergences are cancelled in (21) after substituting $\nu_0 = \nu Z_{\nu}$. This determines Z_{ν} up to an UV finite contribution, which is fixed by the choice of the renormalization scheme. In the MS scheme all the renormalization constants have the form: $1 +$ *poles in* ε, η *and their linear combinations*. In contrast to the rapid-change model, where only one-loop diagram exists (it is related to the fact that all higher-order loop diagrams contain at least one closed loop which is built on by only retarded propagators, thus are automatically equal to zero), in the case with finite correlations in time of the velocity field, higher-order corrections are nonzero. In the two-loop approximation the self-energy operator Σ is defined by diagrams which are shown in figure 2.

As was already mentioned, in our calculations we can put $\eta = 0$. This possibility essentially simplifies the evaluations of all quantities [18, 22].

Two-loop calculations of divergent parts of diagrams in figure 2 give the renormalization constant Z_{ν} and anomalous dimension γ_{ν} (18) in the form:

$$Z_{\nu} = 1 + \frac{g}{\varepsilon}\mathcal{A} + \frac{g^2}{\varepsilon}\mathcal{B} + \frac{g^2}{\varepsilon^2}\mathcal{C}, \qquad \gamma_{\nu} = -2(g\mathcal{A} + 2g^2\mathcal{B}). \tag{22}$$

Here $\mathcal{A} = (1-d)/(2d(1+u))$ is the one-loop contribution to the constant Z_ν and anomalous dimension γ_ν and the two-loop ones are

$$\mathcal{B} = \frac{(d-1)(d+u)}{4d^2(d+2)(1+u)^5} \cdot {}_2F_1\left(1, 1; 2+\frac{d}{2}; \frac{1}{(1+u)^2}\right) - \frac{\pi\rho^2}{36(1+u)^3} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \frac{1}{(1+u)^2}\right), \quad (23)$$

$$\mathcal{C} = -\frac{(d-1)^2}{d^2}\frac{1}{8(1+u)^3},$$

where

$${}_2F_1(a, b, c, z) = 1 + \frac{ab}{c \cdot 1}z + \frac{a(a+1)b(b+1)}{c(c+1)\cdot 1 \cdot 2}z^2 + \cdots$$

represents the hypergeometric function. We substitute $d = 3$ in the helical part (proportional to the ρ), but for completeness we remain the d-dependence in the non-helical one. In addition we have introduced the new notation $g = \mathrm{g}S_d/(2u(2\pi)^d)$ ($S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the d-dimensional sphere).

From the expressions (18)–(20) and (22) we are able to find and classify all fixed points g_*, u_* which satisfy equations:

$$\beta_g(g_*, u_*) = \beta_u(g_*, u_*) = 0. \quad (24)$$

To investigate the infrared stability of a fixed point it is enough to analyse the eigenvalues of the 2×2 matrix Ω of first derivatives: $\Omega_{ij} = \partial\beta_{g_i}\big/\partial g_j (g_i \equiv g, u)$. The anomalous scaling is governed by the infrared stable fixed points, i.e., those for which both eigenvalues Ω_1, Ω_2 are non-negative.

Classification and detailed analysis of all fixed points, determination of region of their stability and influence of helicity will be presented elsewhere. Here we confine ourselves to the most interesting IR stable fixed point, where both parameters g_*, u_* acquire nontrivial values at $\eta = \varepsilon$:

$$g_* = \left((g_*^{(1)} + \left(g_*^{(2)} + g_*^{(3)}\rho^2\right)\varepsilon\right)\varepsilon, \qquad g_*^{(1)} = \frac{3}{2}(1+u_*),$$
$$g_*^{(2)} = \frac{3(3+u_*)}{20(1+u_*)^2} \cdot {}_2F_1\left(1, 1; \frac{7}{2}; \frac{1}{(1+u_*)^2}\right), \quad (25)$$
$$g_*^{(3)} = -\frac{3\pi}{8} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \frac{1}{(1+u_*)^2}\right).$$

Actually, equation (25) represents a line of fixed points in $g-u$ plane. The competition between helical and non-helical terms appears which yields a nontrivial restriction for the fixed point values of variable u to have positive fixed values for variable g.

The first eigenvalue of the stability matrix Ω_{ij} vanishes and the second one Ω_2 is

$$\Omega_2 = \frac{2+u_*}{1+u_*}\varepsilon + \frac{\varepsilon^2}{140(1+u_*)^4}\left[\frac{8u_*(3+u_*)}{(1+u_*)^2}{}_2F_1\left(2, 2; \frac{9}{2}; \frac{1}{(1+u_*)^2}\right) + 14(u_*(3+u_*)-6)\right.$$
$$ {}_2F_1\left(1, 1; \frac{7}{2}; \frac{1}{(1+u_*)^2}\right) + 7\pi\rho^2\left(10(1+u_*)^2\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \frac{1}{(1+u_*)^2}\right)\right.$$
$$\left.\left. - u_*\left(\frac{3}{2}, \frac{3}{2}; \frac{7}{2}; \frac{1}{(1+u_*)^2}\right)\right)\right] \quad (26)$$

with the nontrivial helical part which plays an important role in determination of the region of the IR stability of the fixed point.

4. Effective diffusivity

One of the interesting object from the theoretical as well as experimental point of view is the so-called effective diffusivity $\bar{\nu}$. In this section let us briefly investigate the effective diffusivity $\bar{\nu}$, which replaces the initial molecular diffusivity ν_0 in (1) due to the interaction of the scalar field θ with the random velocity field $\mathbf{v}$. The molecular diffusivity ν_0 governs exponential damping in time of all fluctuations in the system in the lowest approximation, which is given by the propagator (response function) (8). Analogously, the effective diffusivity $\bar{\nu}$ governs exponential damping of all fluctuations described by the full response function, which is defined by the Dyson equation (21). Its explicit expression can be obtained by the RG approach. In accordance with general rules of the RG (see, e.g. [13]) all principal parameters of the model g_0, u_0 and ν_0 are replaced by their effective (running) counterparts, which satisfy the RG flow equations

$$s\frac{\mathrm{d}\bar{g}}{\mathrm{d}s} = \beta_g(\bar{g},\bar{u}), \qquad s\frac{\mathrm{d}\bar{u}}{\mathrm{d}s} = \beta_u(\bar{g},\bar{u}) \qquad s\frac{\mathrm{d}\bar{\nu}}{\mathrm{d}s} = -\bar{\nu}\gamma_\nu(\bar{g},\bar{u}) \tag{27}$$

with initial conditions $\bar{g}|_{s=1} = g$, $\bar{u}|_{s=1} = u$, $\bar{\nu}|_{s=1} = \nu$. Here $s = k/\mu$, β and γ functions are defined in (18)–(20) and the running parameters $\bar{g}$, $\bar{u}$ and $\bar{\nu}$ clearly depend on the variable s. Due to special form of β-functions (19), (20) we are able to solve the last equation (27) analytically. Using the first equation (27) and (19) one immediately rewrites the equation for effective diffusivity in the form

$$\frac{\mathrm{d}\bar{\nu}}{\bar{\nu}} = \frac{\gamma_\nu}{2\varepsilon+\eta-3\gamma_\nu}\frac{\mathrm{d}\bar{g}}{\bar{g}} \tag{28}$$

which can be easily integrated. Using initial conditions the solution acquires the form:

$$\bar{\nu} = \left(\frac{g\nu^3}{\bar{g}s^{2\epsilon+\eta}}\right)^{1/3} = \left(\frac{D_0}{\bar{g}k^{2\epsilon+\eta}}\right)^{1/3}. \tag{29}$$

We emphasize that the above solution is exact, i.e., the exponent $2\epsilon+\eta$ is exact too. However, in the infrared region $k \ll \mu \sim l^{-1}$, $\bar{g} \to g_*$, which can be calculated only perturbatively. In the two-loop approximation $g_* = g_*^{(1)}\varepsilon + (g_*^{(2)} + g_*^{(3)})\varepsilon^2$ and after the Taylor expansion of $g_*^{1/3}$ in (29) we obtain

$$\bar{\nu} \approx \nu_*\left(\frac{D_0}{g_*^{(1)}\varepsilon}\right)^{1/3} k^{-\frac{2\epsilon+\eta}{3}}, \qquad \nu_* \equiv 1 - \frac{(g_*^{(2)}+g_*^{(3)})\varepsilon}{3g_*^{(1)}}. \tag{30}$$

Remind that for Kolmogorov values $\varepsilon = \eta = 4/3$ the exponent in (30) becomes equal to $-4/3$. Let us estimate the contribution of helicity to the effective diffusivity in the fixed point (25). Taking into account at this point $\varepsilon = \eta((2\varepsilon+\eta)/3 = \varepsilon)$ and using the expression (25) the amplitude ν_* in (30) acquires the following form:

$$\nu_* = 1 - \varepsilon\Bigg[\left(\frac{(3+u_*)}{30(1+u_*)^3}\cdot {}_2F_1\left(1,1;\frac{7}{2};\frac{1}{(1+u_*)^2}\right)\right. \\ \left. - \frac{\pi\rho^2}{12(1+u_*)}\cdot {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{(1+u_*)^2}\right)\right)\Bigg]. \tag{31}$$

In figure 3, the dependence of the ν_* on the helicity parameter ρ and the IR fixed point u_* for the Kolmogorov value of parameter ε is shown . As one can see from these figures when $u_* \to \infty$ (the rapid change model limit) the two-loop corrections to $\nu_* = 1$ are vanishing. Such behaviour is related to the fact that within the rapid change model there are no two and higher loop corrections at all. On the other hand, the largest two-loop corrections to the ν_* are given in the frozen velocity field limit ($\nu_* \to 0$).

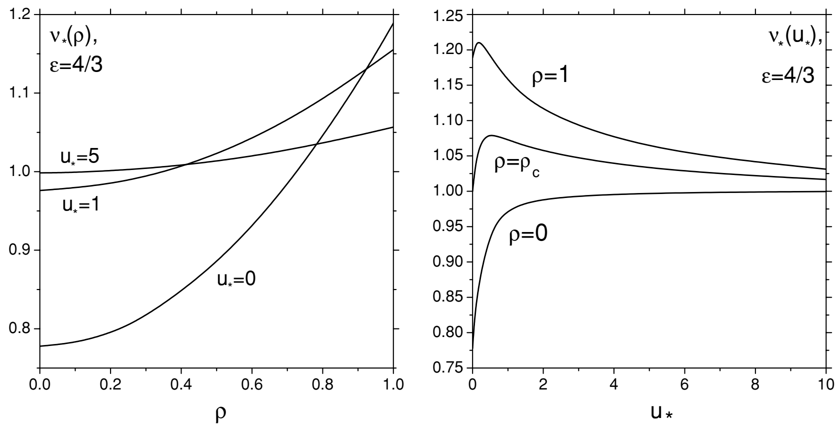

Figure 3. (Left) The dependence of ν_* on the helicity parameter ρ for definite IR fixed point values u_* of the parameter u. (Right) The dependence of ν_* on the IR fixed point u_* for the concrete values of the helicity parameter ρ. The value $\rho_c = 4/\sqrt{3}$. It is a special value related to the analysis of the stability of the scaling regime which is not discussed here.

Finally, let us analyse time behaviour of the retarded response function $G \equiv \langle\theta\theta'\rangle$ in the limit $t \to \infty$.

In frequency–wave vector representation $G(\omega, \mathbf{p})$ satisfies the Dyson equation (21). the self-energy operator Σ is expressed via multi-loop Feynman graphs and can be calculated perturbatively. We have found its divergent part up to the two-loop approximation and calculated its finite part with the one-loop precision.

Using the Dyson equation we find the response function in the time–wave vector representation:

$$G(t, \mathbf{p}) = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t} G(\omega, \mathbf{p}) = \int \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{-\mathrm{i}\omega + \nu_0 p^2 - \Sigma(\omega, \mathbf{p})}. \tag{32}$$

In the lowest approximation $\Sigma(\omega, \mathbf{p}) = 0$; thus the integral can be easily calculated: $G_0(t, \mathbf{p}) = \theta(t)\exp(-\mathrm{i}\omega_r t)$. Here $\theta(t)$ denotes the usual step function and ω_r is a residuum in complex plain ω in point $-\mathrm{i}\nu_0 p^2$. According to [47], where analogical problems have been analysed for turbulent viscosity, we suppose that this situation remains the same for the full response function G; i.e., the leading contribution to its asymptotic behaviour for $t \to \infty$ is determined by the residuum $\omega = \omega_r$, which corresponds to the smallest root of the dispersion relation

$$G^{-1}(\omega, \mathbf{p}) = -\mathrm{i}\omega_r + \nu_0 p^2 - \Sigma(\omega_r, \mathbf{p}) = 0. \tag{33}$$

It is advantageous to rewrite the last relation in the dimensionless form:

$$1 - z - I(1, z) = 0, \qquad z \equiv \frac{\mathrm{i}\omega_r}{\nu_0 p^2}, \qquad I(1, z, g) \equiv \frac{\Sigma(\omega, \mathbf{p})}{\nu_0 p^2}, \tag{34}$$

which after renormalization can be rewritten in the fixed point g_* (25) as follows:

$$1 - z_* - I_* = 0, \qquad z_* \equiv \frac{\mathrm{i}\omega_r}{\bar{\nu} p^2}, \tag{35}$$

where $\bar{\nu}$ is effective diffusivity (30) and $I_* \equiv I_*(1, z_*, g_*)$ is the renormalized (finite) part of the dimensionless self-energy operator I at the fixed point g_*.

Hence decay law $G_0(t, \mathbf{p}) \sim \exp(-\nu_0 p^2 t)$ is changed into

$$G(t, \mathbf{p}) \sim \exp(-\mathrm{i}\omega_r t) = \exp(-z_* \bar{\nu} p^2 t) \qquad t \to \infty. \tag{36}$$

To find the residuum ω_r (or, equivalently, z_*) it is necessary to calculate quantity I_*. In the one-loop (linear in ε) approximation it can be written in the form:

$$I_* = -g_* \int_{-1}^{1} (1 - x^2)^{\frac{d-1}{2}} \, \mathrm{d}x \, \mathcal{I} \tag{37}$$

with

$$\mathcal{I} = \int_0^\infty \mathrm{d}k \left[\frac{k}{1 - z_* + (1 + u_*)k^2 - 2kx} - \frac{\theta(k-1)}{(1 + u_*)k} \right]. \tag{38}$$

Generally, the root z_* can be complex and in the one-loop approximation it has the form

$$z^* = z_1^* + \mathrm{i}z_2, \qquad z_1^* = 1 + x_1\varepsilon, \qquad z_2^* = x_2\varepsilon. \tag{39}$$

With our guaranteed precession I_* is linear in ε, therefore on the first sight it seems that in the last integral it is enough to take $z_* = 1$ ($g_* \sim \varepsilon$), but, actually, for its correct calculation we need to remain the imaginary part $x_2\varepsilon$. Then the integral (38) can be easily calculated by means of Sokhotsky's formula:

$$\lim_{\varepsilon \to 0^+} \frac{1}{y \pm \mathrm{i}\epsilon} = \mp \mathrm{i}\pi\delta(y) + P\left(\frac{1}{y}\right), \tag{40}$$

where $P\left(\frac{1}{y}\right)$ denotes the principal value of the integral. Integration over the angle x gives the final result for the dimensionless self-energy operator I_*:

$$I_* = -\frac{g_*}{(1 + u_*)} \left(\frac{\sqrt{\pi}\Gamma(d+1)\left(\gamma + \psi\left(1 + \frac{d}{2}\right) + 2\ln|1 + u_*|\right)}{d 2^d \Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d}{2} + 1\right)} \pm \mathrm{i}\pi \frac{d-1}{2d} \right), \tag{41}$$

where γ is Euler's constant and $\psi(z)$ is the digamma function defined as $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Successful calculation of integral I allows one to determine the residue z_* (39). Comparison of real and complex parts of both sides of (35) gives the following terms in real space $d = 3$ and in the fixed point $g_* = 3(1 + u_*)/2$ (see (25))

$$x_1 = \frac{8}{3} + 2\ln\frac{1 + u_*}{2}, \qquad x_2 = \pm\frac{\pi}{2}. \tag{42}$$

Due to the existence of two complex conjugate values z_* the response function $G(t, p^2)$ can be written in the asymptotic limit $t \to \infty$ in the following final form:

$$G(t, p^2) \cong \mathrm{e}^{-\nu_{\mathrm{eff}} p^{2-\varepsilon} t} \sin(\nu_f p^{2-\varepsilon} t), \tag{43}$$

where

$$\nu_{\mathrm{eff}} \equiv \left[1 - \varepsilon\left[\frac{8}{3} + 2\ln\frac{1 + u_*}{2} + \frac{(3 + u_*)}{30(1 + u_*)^3} \cdot {}_2F_1\left(1, 1; \frac{7}{2}; \frac{1}{(1 + u_*)^2}\right) - \frac{\pi\rho^2}{12(1 + u_*)} \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \frac{1}{(1 + u_*)^2}\right)\right]\right] \left(\frac{2D_0}{3(1 + u_*)\varepsilon}\right)^{1/3}$$
$$\nu_f \equiv \frac{\pi\varepsilon}{2} \left(\frac{2D_0}{3(1 + u_*)\varepsilon}\right)^{1/3}. \tag{44}$$

It is clear that the exponential damping is accompanied by the oscillations.

5. Conclusion

We have studied the advection of a scalar field by a turbulent flow in the framework of the extended Kraichnan model and investigated the influence of helicity on anomalous scaling, stability of asymptotic regimes and effective diffusivity. This investigation is useful for understanding the efficiency of simplified models to study the real turbulent motions by means of modern theoretical methods, including the renormalization group approach. Actually, we performed two-loop calculations of the divergent parts of the Feynman graphs, which are necessary to achieve multiplicative renormalization of an equivalent field theoretic model. We have shown that anomalous scaling, which is typical of the Kraichnan model and its numerous extensions [22, 48], is not violated by the inclusion of helicity in the incompressible fluid. On the other hand, stability of asymptotic regimes, values of fixed RG points and turbulent diffusivity strongly depend on the amount of helicity. It can be easily seen from (31) that helicity enlarges the turbulent diffusivity and high order contributions lead to the appearance of oscillations in the response function (32).

Acknowledgments

MH is thankful to N V Antonov and L Ts Adzhemyan for discussion. The work was supported in part by VEGA grant 2/6193/26 of the Slovak Academy of Sciences, by the Science and Technology Assistance Agency under contract No. APVT-51-027904, by grant RFFI-RFBR 05-05-64735, by grant RFFI-RFBR 05-02-17603, and by COFIN 2003 'Sistemi Complessi e Problemi a Molti Corpi.'.

References

[1] Monin A S and Yaglom A M 1975 *Statistical Fluid Mechanics* vol 2 (Cambridge, MA: MIT Press)
[2] McComb W D 1990 *The Physics of Fluid Turbulence* (Oxford: Clarendon)
[3] Frisch U 1995 *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge: Cambridge University Press)
[4] Falkovich G, Gawčdzki K and Vergassola M 2001 *Rev. Mod. Phys.* **73** 913
[5] Obukhov A M 1949 *Izv. Akad. Nauk SSSR: Geogr. Geofiz.* **13** 58
[6] Kraichnan R H 1968 *Phys. Fluids* **11** 945
[7] Holzer M and Siggia E D 1994 *Phys. Fluids* **6** 1820
[8] Kraichnan R H 1994 *Phys. Rev. Lett.* **72** 1016
[9] Gawędzki K and Kupiainen A 1995 *Phys. Rev. Lett.* **75** 3834
[10] Pumir A 1997 *Europhys. Lett.* **37** 529
Pumir A 1998 *Phys. Rev.* E **57** 2914
[11] Majda A 1993 *J. Stat. Phys.* **73** 515
[12] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)
[13] Vasil'ev A N 2004 *Quantum-Field Renormalization Group in the Theory of Critical Phenomena and Stochastic Dynamics* (New York: Chapman and Hall)
[14] Adzhemyan L Ts, Antonov N V and Vasil'ev A N 1998 *Phys. Rev.* E **58** 1823
[15] Adzhemyan L Ts, Antonov N V and Vasil'ev A N 1999 *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (London: Gordon and Breach)
[16] Adzhemyan L Ts, Antonov N V, Barinov V A, Kabrits Yu S and Vasil'ev A N 2001 *Phys. Rev.* E **63** 025303
Adzhemyan L Ts, Antonov N V, Barinov V A, Kabrits Yu S and Vasil'ev A N 2001 *Phys. Rev.* E **64** 019901(E)
Adzhemyan L Ts, Antonov N V, Barinov V A, Kabrits Yu S and Vasil'ev A N 2001 *Phys. Rev.* E **64** 056306
[17] Adzhemyan L Ts, Antonov N V, Hnatich M and Novikov S 2000 *Phys. Rev.* E **63** 016309
[18] Antonov N V 1999 *Phys. Rev.* E **60** 6691
Antonov N V 2000 *Physica* D **144** 370
[19] Antonov N V, Lanotte A and Mazzino A 2000 *Phys. Rev.* E **61** 6586
[20] Adzhemyan L Ts, Antonov N V, Mazzino A, Muratore-Ginanneschi P and Runov A V 2001 *Europhys. Lett.* **55** 801

[21] Shraiman B I and Siggia E D 1994 *Phys. Rev.* E **49** 2912
Shraiman B I and Siggia E D 1996 *Phys. Rev. Lett.* **77** 2463

[22] Adzhemyan L Ts, Antonov N V and Honkonen J 2002 *Phys. Rev.* E **66** 036313

[23] Vainstein S I, Zel'dovich Ya A and Ruzmaykin A A 1980 *Turbulent Dynamo in Astrophysics* (Moskva: Nauka) (in Russian)

[24] Moffatt H K 1978 *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge: Cambridge University Press)

[25] Etling D 1985 *Beitr. Phys. Atmos.* **58** 88

[26] Ponomarev V M, Khapaev A A and Chkhetiani O G 2003 *Izv.: Atmos. Ocean. Phys.* **39** 391

[27] Brissaud A, Frisch U, Leorat J, Lesieur M and Mazure A 1973 *Phys. Fluids* **16** 1363

[28] Moiseev S S and Chkhetiani O G 1996 *JETP* **83** 192

[29] Chkhetiani O G 1996 *JETP Lett.* **63** 808

[30] Kraichnan R H 1973 *J. Fluid Mech.* **59** 745

[31] Chen Q, Chen S and Eyink G L 2003 *Phys. Fluids* **15** 361

[32] Pouquet A, Fournier J D and Sulem P L 1978 *J. Phys. Lett.* **39** 199

[33] Zhou Ye 1990 *Phys. Rev.* A **41** 5683

[34] Drummond S T, Duane S and Horgan R P 1984 *J. Fluid Mech.* **138** 75

[35] Knobloch E 1977 *J. Fluid Mech.* **83** 129

[36] Lipscombe T C, Frencel A L and ter Haar D 1991 *J. Stat. Phys.* **63** 305

[37] Belyan A V, Moiseev S S, Golbraih E I and Chkhetiani O G 1998 *Physica* A **258** 55

[38] Dolginov A Z and Silantiev N A 1987 *JETP* **93** 159

[39] Dean D S, Drummond I T and Horgan R P 2002 *Phys. Rev.* E **63** 61205

[40] Eyink G 1996 *Phys. Rev.* E **54** 1497

[41] Bouchaud J P and Georges A 1990 *Phys. Rep.* **195** 127

[42] Honkonen J, Pis'mak Yu M and Vasil'ev A N 1989 *J. Phys. A: Math. Gen.* **21** 835

[43] Martin P C, Siggia E D and Rose H A 1973 *Phys. Rev.* A **8** 423

[44] Janssen H K 1976 *Z. Phys.* B **23** 377

[45] Bausch R, Janssen H K and Wagner H 1976 *Z. Phys.* B **24**

[46] De Dominicis C 1976 *J. Physique* **37** (Suppl. C1) 247

[47] Adzhemyan L V and Adzhemyan L Ts 2003 *Vest. Sankt Peterburgskoho Univ. Ser. Fiz. Chemia* **4** 94 (in Russian)

[48] Antonov N V, Hnatich M, Honkonen J and Jurcisin M 2003 *Phys. Rev.* E **68** 046306